On The Generalized Completeness

Buhyeon Kang[1]
[1]*Department of Applied Mathematics, Seokyeong University, Seoul 02713, Republic of Korea*

Abstract- We introduced the concept of the generalized limit (or, ε_0-limit) of multi-valued sequences in [2]. The concept of the generalized limit is an extension of the concept of the usual limit of single-valued sequence. This concept of the generalized limit is also an extension of the concept of the approximation. In this paper, we introduce a concept of the generalized completeness using these concepts of the generalized limits and study some properties relating to these concepts.

Keywords –multi-valued function; generalization of limit; ε_0-limit; generalized completeness.

I. INTRODUCTION

The concept of the generalized limit is an extension of the concept of the usual limit of single-valued sequence and function.Andthe concept of the generalized limit isan extension of the concept of the approximation. We need sometimes the limits of multi-valued sequences and functions and the approximation of the unspecified number. In this section, we study briefly those concepts of the generalize limit and some results which we need later.

Definition 1.1. Let $\{x_n\}$ be a vector-valued and multi-valued infinite sequence of elements of R^m. And let $\varepsilon_0 \geq 0$ be any, but fixed, non-negative real number. If a set S satisfies the following condition, we call that theε_0 generalized limit (or ε_0-limit) of $\{x_n\}$ as n goes to ∞ is S, and we denote it by $\varepsilon_0\text{-}\lim_{n\to\infty} x_n = S$:S is the set of all the vectors$\alpha \in R^m$ satisfying the condition

$$\forall \epsilon > \epsilon_0, \exists K \in N \text{ s.t. } (\forall n \in N) n \geq K, (\forall x_n) \Rightarrow \|x_n - \alpha\| < \epsilon.$$

If the set S in the definition above is not empty we say that $\{x_n\}$ is anε_0-convergent sequence or ε_0-converges to S. We also define thatany member $\alpha \in S$ is an approximate value of the generalized limit of $\{x_n\}$ with the limit of the errorε_0. Then we can regard $\alpha \in S$ as the approximate value of the limit of $\{x_n\}$ whether $\{x_n\}$ converges in the usual sense or not.

Definition 1.2. For a multi-valued infinite sequence $\{x_n\}$ of vectors in R^m, we call that $\{x_n\}$ is ultimately bounded if and only if there exist real numbers K and M such that $(\forall n \in N) n \geq K, \forall x_n \Rightarrow \|x_n\| \leq M$.

Lemma 1.3. (Representation) Let $\{x_n\}$ be a vector-valued and multi-valued infinite sequence. And let$\epsilon_0 \geq 0$be a non-negative real number. Suppose that $\{x_n\}$ is ultimately bounded. Ifhave $\varepsilon_0\text{-}\lim_{n\to\infty} x_n = S$then S isa convex and compact subset of R^m such that S= $\cap \{\overline{B}(\alpha, \varepsilon_0) : \alpha \in SSL\}$. Here $\overline{B}(\alpha, \varepsilon_0)$ denotes the closed ball $\overline{B}(\alpha, \epsilon_0) = \{x \in R^m : \|x - \alpha\| \leq \epsilon_0\}$and

$SSL = SSL(\{x_n\}) = \{\alpha \in R^m | \exists \{x_{n_k}\} \leq \{x_n\} \text{ s.t. } \lim_{k\to\infty} x_{n_k} = \alpha\}$

and$\{x_{n_k}\} \leq \{x_n\}$ means that $\{x_{n_k}\}$ is a single-valued subsequence of $\{x_n\}$.

Proof. ($\subseteq$) Let any elements $\beta \in S$ and $\alpha \in SSL$ begiven. Then

$$\forall \epsilon > \epsilon_0, \exists K_1 \in N \text{ s.t. } (\forall n \in N) n \geq K_1, (\forall x_n) \Rightarrow \|x_n - \beta\| < \epsilon_0 + \frac{\epsilon - \epsilon_0}{2}.$$

Since $\alpha \in SSL$, there exists a single-valued and convergent subsequence$\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = \alpha$.Thus we have

$$\forall \epsilon > \epsilon_0, \exists K_2 \in N \text{ s.t. } (\forall k \in N) k \geq K_2 \Rightarrow \|x_{n_k} - \alpha\| < \frac{\epsilon - \epsilon_0}{2}.$$

.If we take a natural number $K = \max\{K_1, K_2\}$ then we have

$$\|\beta - \alpha\| = \|\beta - x_{n_K} + x_{n_K} - \alpha\| \leq \|\beta - x_{n_K}\| + \|x_{n_K} - \alpha\| < \epsilon_0 + \frac{\epsilon - \epsilon_0}{2} + \frac{\epsilon - \epsilon_0}{2} = \epsilon.$$

Since $\varepsilon > \varepsilon_0$ was arbitrary, we have $\|\beta - \alpha\| \leq \varepsilon_0$. That is, $\beta \in \overline{B}(\alpha, \varepsilon_0)$. Since $\alpha \in SSL$ was arbitrary, we have $\beta \in \cap \{\overline{B}(\alpha, \varepsilon_0) : \alpha \in SSL\}$. Since $\beta \in S$ was also arbitrary, we have$S \subseteq \cap \{\overline{B}(\alpha, \varepsilon_0) : \alpha \in SSL\}$. ($\supseteq$) It isobvious thatS $\neq$ Rm since {xn} is ultimately bounded. In order to prove the opposite inclusion, let β $\notin$ S be any member of Rm − S $\neq$ ∅. Then we have

$\exists \epsilon_1 > \epsilon_0 \; s.t. (\forall k \in N, \exists n_k \in N, \exists x_{n_k} \; s.t. \left\|x_{n_k} - \beta\right\| \geq \epsilon_1)$.

Since {x_{n_k}} is ultimately bounded, {x_{n_k}} is a bounded sequence in Rm.Hence there is a convergent subsequence {$x_{n_{k_p}}$} of {x} by the BolzanoWeierstrass theorem. Thus we may assume that $lim_{p\to\infty} x_{n_{k_p}}$= α0 for some α0 ∈Rm. Then we have, for such an $\epsilon_1 > \epsilon_0$,

$\exists K \in N \; s.t. \; \forall p \geq K \Rightarrow ||x_{n_{k_p}} - \alpha_0|| < \frac{\epsilon_1 - \epsilon_0}{2}$.

Hence we have

$||\beta - \alpha_0|| = \left\|\beta - x_{n_{k_K}} + x_{n_{k_K}} - \alpha_0\right\| \geq \left\|\beta - x_{n_{k_K}}\right\| - \left\|x_{n_{k_K}} - \alpha_0\right\| > \epsilon_1 - \frac{\epsilon - \epsilon_0}{2} = \frac{\epsilon + \epsilon_0}{2}$.

Since $\frac{\epsilon_1 + \epsilon_0}{2} > \epsilon_0$, this implies that$\beta \notin \overline{B}(\alpha_0, \epsilon_0)$. Since α0 ∈SSL, this implies that$\beta \notin \cap \{\overline{B}(\alpha, \epsilon_0) : \alpha \in SSL\}$.Thus we have $S = \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. On the other hand, since S is the intersection of the closed balls $\overline{B}(\alpha, \epsilon_0)$which are bounded, closed and convex, S is compact and convex inRm.Finally, if S = ∅ then S is clearly convex and compact, and$\bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0) \subseteq S = \emptyset$. □

Corollary 1.4. Let {xn} be a multi-valued infinite sequence of vectors in Rm and SSL = {a} for some vector a ∈Rm. Then we haveε0-limn→∞xn= $\overline{B}(a, \epsilon_0)$for all$\epsilon_0 \geq 0$.

Proof. Since the sub-sequential limit a of {xn} is unique, this corollary follows from the above lemma 1.3. □

Lemma 1.5. Let {xn} be a vector-valued and multi-valued infinite sequence and$\epsilon_0 \geq 0$. Suppose that {xn} is an ultimately bounded sequence. Then the set SSL of all the single-valued sub-sequential limits of {xn} is a non-empty and compact subset of Rm.

Proof. Since the sequence {xn} is ultimately bounded, the set SSL is clearly non-empty and bounded. In order to prove that SSL is aclosed subset of Rm, let any element α ∈SSL be given. If α ∈SSL we are done. If α$\notin$SSL then α must be an accumulation point of the set SSL. Hence there is a single-valued sequence {αn} ⊆SSL such that limn→∞αn = α. Since α1 ∈SSL, there is one value, say x_{n_1}, of the multi-valued term x_{n_1}in {xn} such that $\|x_{n_1} - \alpha_1\| < 1$. And since α2 ∈SSL, there is one value, say x_{n_2}, of the multi-valued term x_{n_2}in {xn} such that $\|x_{n_2} - \alpha_2\| < \frac{1}{2}$ and n2 > n1. By applying those methods, we can inductively choose a single-valued subsequence { x_{n_k} } of {xn} such that $||x_{n_k} -$ αk $|| < \frac{1}{k}$ for all natural number k ∈N. Since $||x_{n_k} - \alpha|| \leq ||x_{n_k} - \alpha_k|| + ||\alpha_k - \alpha||$, if we take the limit on both sides we have limk→∞x_{n_k}= α. Thus we have α ∈SSL which completes the proof. □

II. THE GENERALIZED COMPLETENESS

In this section, we define the concept of the$\epsilon_0$generalized completeness of a set. Note that$\epsilon_0$denotes some fixed non-negative real number.

Definition 2.1. Let {xn} be a multi-valued sequence in Rm. We define that {xn} is an ϵ_0-Cauchy sequence if and only if

$\forall \epsilon > \epsilon_0, \exists K \in N \; s.t. (\forall m, n) m, n \geq K, \forall x_m, \forall x_n \Rightarrow \left\|x_m - x_n\right\| < \epsilon.$

Definition 2.2. Let Sbe any non-empty subset of Rm. Then we define thatS isϵ_0-complete in Rm if and only if ε0-limn→∞xn$\cap S \neq \emptyset$ forany ϵ_0-Cauchysequence {xn} in S.

Lemma 2.3. Let {xn} be an ϵ_0-Cauchy sequence in Rm. If α ∈SSL then α ∈ε0-limn→∞xn.

Proof. Let {xn} be an ϵ_0-Cauchy sequence in Rm. Then we have

$\forall \epsilon > \epsilon_0, \exists K \in N \; s.t. (\forall m, n) m, n \geq K, \forall x_m, \forall x_n \Rightarrow \left\|x_m - x_n\right\| < \epsilon_0 + \frac{\epsilon - \epsilon_0}{2}$

since $\varepsilon_0 + \frac{\varepsilon_1-\varepsilon_0}{2} > \varepsilon_0$. Since α ∈SSL, there is a single-valued and convergent subsequence $\{x_{n_k}\}$ of {xn} such that $\lim_{n\to\infty} x_{n_k} = \alpha$. Now since nk≥ k, we have, by replacing xnto x_{n_k},

$\forall\epsilon > \epsilon_0, \exists K \in N\ s.t. (\forall m,k)\, m,k \geq K, \forall x_m \Rightarrow \left\|x_m - x_{n_k}\right\| < \epsilon_0 + \frac{\epsilon-\epsilon_0}{2}$.

For each fixed term number m and each value of xm, by taking the limit as kgoes to ∞, we have

$\forall\epsilon > \epsilon_0, \exists K \in N\ s.t. (\forall m)\, m \geq K, \forall x_m \Rightarrow \left\|x_m - \alpha\right\| \leq \epsilon_0 + \frac{\epsilon-\epsilon_0}{2} = \frac{\epsilon+\epsilon_0}{2} < \epsilon$.

Hence α ∈ε0-limn→∞xn.□

Corollary 2.4. Let {xn} be an ε_0-Cauchy sequence in Rm. If we denote by hull(SSL) the convex hull of SSL then hull(SSL)$\neq$∅and

hull(SSL) ⊆ε0-limn→∞xn=$\bigcap_{\alpha\in SSL} \overline{B}(\alpha, \varepsilon_0)$.

Proof. This follows from lemmas 1.3, 2.3 and the convex property of the ε_0-limit. □

Lemma 2.5. Let {xn} be an ε_0-Cauchy sequence in Rm. Then the diameter of SSL is less than or equal to ε_0.

Proof. Letα,β ∈SSL be any two elements. Since {xn} is an ε_0-Cauchy sequence in Rm, we have

$\forall\epsilon > \epsilon_0, \exists K \in N\ s.t. (\forall m,n)\, m,n \geq K, \forall x_m, \forall x_n \Rightarrow \|x_m - x_n\| < \epsilon_0 + \frac{\epsilon - \epsilon_0}{2}$

since $\varepsilon_0 + \frac{\varepsilon_1-\varepsilon_0}{2} > \varepsilon_0$. And since α,β ∈SSL, there are two single-valued and convergent subsequences {xmk} and {xnk} of {xn} such that $\lim_{m\to\infty} x_{m_k} = \alpha$ and $\lim_{n\to\infty} x_{n_k} = \beta$. Since mk,nk≥ k, we have

$\forall\epsilon > \epsilon_0, \exists K \in N\ s.t. (\forall k)\, k \geq K \Rightarrow \left\|x_{m_k} - x_{n_k}\right\| < \epsilon_0 + \frac{\epsilon-\epsilon_0}{2}$..

By taking the limit as kgoes to ∞, we have

$\left\|\alpha - \beta\right\| \leq \epsilon_0 + \frac{\epsilon-\epsilon_0}{2} = \frac{\epsilon+\epsilon_0}{2} < \epsilon$.

Since $\varepsilon > \varepsilon_0$ was arbitrary, this implies that $\|\alpha - \beta\| \leq \varepsilon_0$. Hence diam(SSL)$\leq \varepsilon_0$. □

Proposition 2.6. Let {xn} be an ε_0-Cauchy sequence in Rm. Then {xn} is ultimately bounded.

Proof. Since {xn} is an ε_0-Cauchy sequence, we have

$\exists K \in N\ s.t. (\forall m,n)\, m,n \geq K, \forall x_m, \forall x_n \Rightarrow \left\|x_m - x_n\right\| < \epsilon_0 + 1$.

Choosing one value xKof xK, we have

$\exists K \in N\ s.t. (\forall m)\, m \geq K, \forall x_m \Rightarrow \left\|x_m - x_K\right\| < \epsilon_0 + 1$.

Hence we have

$\exists K \in N\ s.t. (\forall m)\, m \geq K, \forall x_m \Rightarrow \left\|x_m\right\| < \epsilon_0 + \|x_K\| + 1$.

Thus {xn} is ultimately bounded. □

Proposition 2.7. Let {xn} be anε_0 −Cauchy sequence in Rm. If $\varepsilon_0 > 0$ and diamSSL({xn})= d then there exists a vector γ ∈ Rm and a positive real number $r \geq (\varepsilon_0 - \frac{\sqrt{3}}{2}d) > 0$such that B(γ,r)∩ hull(SSL) $\neq$∅ and B(γ,r)⊆ε0-limn→∞xn..

Proof. Note that$d \leq \varepsilon_0$ since the diameter of SSL is less than or equal to $\varepsilon_0$by lemma 2.5. If d = 0 then SSL = {γ} is a singleton for some γ ∈Rm.Hencewe have{γ} ⊆B(γ,0) = ε0-limn→∞xnby the corollary 1.4. Suppose that d >0. Then there are two distinct elements x0,y0 ∈SSL such that ||x0 − y0||= d since SSL is compact by the lemma 1.5. By an appropriate rotation and translation of the axes and the origin in the usual Euclidean coordinate system of Rm, we may assume that $x_0 = (-\frac{d}{2}, 0, \cdots, 0)$, $y_0 = (\frac{d}{2}, 0, \cdots, 0)$ and $\frac{x_0+y_0}{2} =$(0,0,⋯ ,0). Then we must have

SSL ⊆B(x0,d) ∩ B(y0,d)

sincediamSSL({xn}) = d. But the equation of the most far boundary from the origin of the intersection of the boundaries ∂B(x0,d) and∂B(y0,d) is given by

$\left(x_1 - \frac{d}{2}\right)^2 + x_2^2 + \cdots + x_m^2 = d^2 = \left(x_1 + \frac{d}{2}\right)^2 + x_2^2 + \cdots + x_m^2$.

That is, we have

$x_1 = 0,\ x_2^2 + \cdots + x_m^2 = \frac{3}{4}d^2$.

Thus the distance between the origin and the boundary of the intersection B(x0,d) ∩ B(y0,d) satisfies that

$\mathrm{dist}(0, \partial[\overline{\mathrm{B}}(x_0, d) \cap \overline{\mathrm{B}}(y_0, d)]) \leq \frac{\sqrt{3}}{2}d$.

Hence we have $\mathrm{SSL} \subseteq \overline{\mathrm{B}}(0, \frac{\sqrt{3}}{2}d)$. Then we also have $\mathrm{hull(SSL)} \subseteq \overline{\mathrm{B}}(0, \frac{\sqrt{3}}{2}d)$ since $\overline{\mathrm{B}}(0, \frac{\sqrt{3}}{2}d)$ is convex. Since $\overline{\mathrm{B}}(0, \frac{\sqrt{3}}{2}d)$ is a ballwhose boundary is not a convex set, this implies that $\mathrm{hull(SSL)} \cap \mathrm{B}(0, \frac{\sqrt{3}}{2}d) \neq \emptyset$ if hull(SSL) is not a singleton. In fact,hull(SSL) is not a singleton since the diameter d of SSL is positive.Now we have proved that

$\mathrm{SSL} \subseteq \overline{\mathrm{B}}(\frac{x_0+y_0}{2}, \frac{\sqrt{3}}{2}d)$ and $\overline{x_0y_0} \cap B(\frac{x_0+y_0}{2}, \frac{\sqrt{3}}{2}d) \neq \emptyset$.

Thus we have

$\overline{\mathrm{B}}\left(\frac{x_0+y_0}{2},\ \epsilon_0 - \frac{\sqrt{3}}{2}d\right) = \bigcap_{\alpha \in \bar{B}(\frac{x_0+y_0}{2}, \frac{\sqrt{3}}{2}d)} \bar{B}(\alpha, \epsilon_0) \subseteq \bigcap_{\alpha \in SSL} \bar{B}(\alpha, \epsilon_0) =$ ε0-limn→∞xn..

By taking $\gamma = \frac{x_0+y_0}{2}$ and $r = \varepsilon_0 - \frac{\sqrt{3}}{2}d$, we have the second result in this proposition. And we have $\overline{x_0y_0} \cap B(\gamma, r) \subseteq \mathrm{B}(\frac{x_0+y_0}{2}, \frac{\sqrt{3}}{2}d)$. Thus we have $\overline{x_0y_0} \cap B(\gamma, r) \subseteq \mathrm{B}(\alpha, \varepsilon_0)$ for all $\alpha \in \mathrm{B}(\frac{x_0+y_0}{2}, \frac{\sqrt{3}}{2}d)$. This givesthe first result in this proposition which completes the proof. □

Theorem 2.8. If D ⊆Rm satisfies $\bigcup_{b \in D} \bar{B}(b, \left(1 - \frac{\sqrt{3}}{2}\right)\varepsilon_0) = R^m$ then D is ε_0-complete.

Proof. At first, assume that $\varepsilon_0 = 0$ and let any 0-Cauchy sequence {xn} be given. Then any single-valued subsequence of {xn} is a Cauchy sequence in the usual sense. Since Rm is complete in the usual sense and {xn} is a 0-Cauchy sequence, the set of all the sub-sequential limits SSL({xn}) must be a singleton. Thus {xn} is a 0-convergent sequence. Suppose that $\varepsilon_0 > 0$. Let any ε_0-Cauchy sequence {xn} in Rm be given. If we set diam(SSL) = d, then, by the proposition above, we have

$\bar{B}(\gamma, \varepsilon_0 - \frac{\sqrt{3}}{2}d)$ ⊆ε0-limn→∞xn.

for some vector γ ∈Rm. Since $d \leq \varepsilon_0$ by lemma 2.5, $\overline{\mathrm{B}}\left(\gamma, \left(1 - \frac{\sqrt{3}}{2}\right)\varepsilon_0\right)$ is a subset of $\bar{B}(\gamma, \varepsilon_0 - \frac{\sqrt{3}}{2}d)$. Thus we have

$\overline{\mathrm{B}}\left(\gamma, \left(1 - \frac{\sqrt{3}}{2}\right)\varepsilon_0\right)$ ⊆ε0-limn→∞xn.

But if $D \cap \overline{\mathrm{B}}\left(\gamma, \left(1 - \frac{\sqrt{3}}{2}\right)\varepsilon_0\right) = \emptyset$ then we have

$$\gamma \notin \bigcup_{b \in D} \overline{\mathrm{B}}\left(b, \{1 - \frac{\sqrt{3}}{2}\}\varepsilon_0\right) = R^m$$

which is a contradiction. Thus we have D ∩ ε0-limn→∞xn≠∅. Therefore, D is ε_0-complete. □

Note that the set R^m is ε_0-complete for any non-negative real number ϵ0 ≥0 by the proposition above since $R^m = \bigcup_{b \in R^m} \bar{B}(b, \left(1 - \frac{\sqrt{3}}{2}\right)\varepsilon_0)$. And if $\varepsilon_0 > 0$ then any dense subset D of Rm in the usual sense is also ε_0-complete since $R^m = \bigcup_{b \in D} \bar{B}(b, \left(1 - \frac{\sqrt{3}}{2}\right)\varepsilon_0)$. In particular, bothQmand Rm − Qmare ε_0-complete. But neither Qmnor Rm − Qmare 0-complete as we know.

Theorem 2.9. Any closed subset D of Rm is ε_0-complete for all $\varepsilon_0 \geq 0$.

Proof. Suppose that D is a closed subset of Rm and let any ε_0-Cauchy sequence {xn} ⊆D be given. By corollary 2.4, we have

SSL ⊆ε0-limn→∞xn.

But the set SSL({xn})≠∅ since {xn} is ultimately bounded by proposition 2.6. Since SSL ⊆$\overline{\mathrm{D}}$, this implies that

∅≠SSL ⊆$\bar{D}$∩ ε0-limn→∞xn .

But we have $\bar{D}$=Dsince D is closed. Thus D is ε_0-complete for all $\varepsilon_0 \geq 0$.□

Corollary 2.10. Let D$\neq\varnothing$be a subset of Rm and a real number$\varepsilon_0 \geq 0$be given. IfD is ε_0-complete then$\overline{D}$ is ε_0-complete. But the converse is not true in general.

Proof. By the theorem just above, it is clear that$\overline{D}$ is ε_0-complete. Now consider the subset D of R given by

$$\mathrm{D} = \{-\frac{1}{n}, 1+\frac{1}{n} : n \in N\}.$$

Then D = D ∪ {0,1} is 1-complete since it is closed. But if we choose a sequence {xn} such that$x_{2n} = -\frac{1}{2n}$ and$x_{2n-1} = 1+\frac{1}{2n-1}$ for each n ∈N then SSL({xn}) = {0,1}. Hence we have

ε0-limn→∞xn=$\bigcap_{\alpha\in\{0,1\}} B(\alpha,1)$= [0,1].

Since D ∩ [0,1] = ∅, D is not 1-complete. □

Theorem 2.11. Any convex subsetD of Rm is ε_0-complete for all positive real number$\varepsilon_0 > 0$.

Proof. Suppose that D is a convex subset of Rm. Since $\emptyset$is ε_0-complete, we may assume that D$\neq\varnothing$. And let any ε_0-Cauchy sequence{xn} ⊆D be given. Since {xn} is also an ε_0-Cauchy sequence in D which is ε_0-complete by theorem 2.9 and D is also convex, we have

∅$\neq$hull(SSL) ⊆D ∩ε0-limn→∞xn .

But if D ∩hull(SSL({xn})) $\neq\varnothing$ then we are done since the intersectionofDand the ε_0-limit of {xn} is not an empty set. Now suppose that D∩hull(SSL({xn})) = ∅. Then hull(SSL) is a subset of the derived set D', the set of all the accumulation points of D. That is, hull(SSL) ⊆$D'-D$. Hence hull(SSL) is a subset of the boundary ∂D of D. By proposition 2.7, there are some vector γ and some real number r >0 such that hull(SSL) ∩ B(γ,r) $\neq\varnothing$ and B(γ,r) ⊆ε0-limn→∞xn=$\bigcap_{\alpha\in SSL} B(\alpha,\varepsilon_0)$.

Now choose a point β ∈hull(SSL) ∩ B(γ,r) $\neq\varnothing$. Then β ∈$D'-D$. Hence there is an element β0 ∈D such that β0 ∈B(γ,r) since B(γ,r) is an open set containing the accumulation point β. Thus β0 ∈D ∩ ε0-limn→∞xnwhich completes the proof. □

Note that the convex subset of Rm is not 0-complete in general.

Proposition 2.12. (1) The union of the ε_0-complete subsets does not need to be ε_0-complete. (2) The intersection of the ε_0-complete subsets does not need to be ε_0-complete.

Proof. (1) Let$D_1 = \{-\frac{1}{n} : n \in N\}$and $D_2 = \{1+\frac{1}{n} : n \in N\}$.In order to prove that D1 is 1-complete, let any 1-Cauchy sequence {xn} ⊆D1 be given. Then SSL({xn}) $\neq\varnothing$ and SSL ⊆D1∪{0}. Hence we have

[−1,0] ⊆$\bigcap_{\alpha\in D\cup\{0\}} B(\alpha,1)$⊆$\bigcap_{\alpha\in SSL} B(\alpha,1)$=1-limn→∞xn.

Thus the intersection of D1 and the 1-limit of {xn} is not an empty set. HenceD1 is 1-complete. Since the diameter of D2 is 1, we can prove by the same method that D2 is also 1-complete. But the union

$$\mathrm{D}_1 \cup \mathrm{D}_2 = \{-\frac{1}{n}, 1+\frac{1}{n} : n \in N\}$$

is not 1-complete as in the proof of corollary 2.10. (2) Let $D_1 = \{-\frac{1}{n}, 0, 1+\frac{1}{n} : n \in N\}$ and $D_2 = \{-\frac{1}{n}, 1, 1+\frac{1}{n} : n \in N\}$. In order toprove that D1 is 1-complete, let any 1-Cauchy sequence {xn} ⊆D1 be given. Since the diameter of SSL satisfies the inequality diam(SSL) ≤ 1, the following three cases occur.

$\emptyset \neq \mathrm{SS}$

$\emptyset \neq \mathrm{SSL} = \{-\frac{1}{n},$

$\emptyset \neq \mathrm{SSL} = \{1+\frac{1}{n},$.

(i) If SSL = {0,1} then D1 ∩ 1 − limxn= D1 ∩ [0,1] = {0}$\neq\varnothing$. (ii) If$\mathrm{SSL} \subseteq \{-\frac{1}{n}, 0 : n \in N\}$then D1 ∩ 1 − limxn= {$-\frac{1}{n}$,0:n ∈N} $\neq\varnothing$.(iii) If $\mathrm{SSL} \subseteq \{1+\frac{1}{n}, 1 : n \in N\}$ then D1 ∩ 1 − limxn= {$1+\frac{1}{n}$: n ∈N} $\neq\varnothing$. Therefore, D1 is 1-

complete. On the other hand, we can prove by the same method that D2 is also 1-complete. But the intersection

$D_1 \cap D_2 = \{-\frac{1}{n}, 1+\frac{1}{n} : n \in N\}$

is not 1-complete as in the proof of (1). □

Definition 2.13. Let V = {v1,v2,⋯ ,vm+1} be a subset of Rm. We define that V is an m-dimensional ε_0-tetrahedral vertex if and only if $\|v_i - v_j\| = \varepsilon_0$ for all i≠j.And we call the set $\bigcap_{1\le k\le m+1} B(v_k, \varepsilon_0)$ (resp. $\bigcap_{1\le k\le m+1} \overline{B}(v_k, \varepsilon_0)$) an m-dimensional ε_0-tetrahedral open(resp. closed)ball and denote by $T_m(V, \varepsilon_0)$(resp. $\overline{T_m}(V, \varepsilon_0)$).

Proposition 2.14. Let $\varepsilon_0 > 0$ be a positive real number and a subsetD of R^m be ε_0-complete. Then $D \cap \overline{T_m}(V, \varepsilon_0) \neq \emptyset$ for each m-dimensional ε_0-tetrahedral vertex V such that V ⊆ $D' - D$.

Proof. Suppose that $D \cap \overline{T_m}(V, \varepsilon_0) = \emptyset$ for some m-dimensional ε_0-tetrahedral vertex V such that V ⊆ $D' - D$. Let V = {v1,v2,⋯ ,vm+1}. Then, for each 1 ≤ k ≤ m+1, there is a sequence $\{v_{k_p}\}$ in D such that $\lim_{p\to\infty} v_{k_p}$ = vk. For each natural number p ∈N, there are non-negativeintegers u,r∈N ∪ {0} such that p = (m + 1)u + r and 0 ≤ r < m + 1.Now let's define the sequence {xp} as follows.

xp= v(r+1)p, p = (m + 1)u + r, u,r ∈N∪{0}, 0 ≤ r < m + 1

for each natural number p∈N. Since $\lim_{p\to\infty} v_{k_p} = v_k$ for each 1 ≤k ≤ m + 1, we have

$$\forall \epsilon > \epsilon_0, \exists K \in N\ s.t. \forall p \ge K, 1 \le k \le (m+1) \Rightarrow \left\| v_{k_p} - v_k \right\| < \frac{\epsilon - \epsilon_0}{2}.$$

In order to show that {xp} is an ε_0-Cauchy sequence, for each positive real number $\varepsilon > \varepsilon_0$, let any natural number p,q ≥ K be given. Then, by the Euclidean division theorem, we have

∃u,r∈N ∪ {0} s.t. p = (m + 1)u + r and 0 ≤ r < m + 1

and

∃t,s∈N ∪ {0} s.t. q = (m + 1)t + s and 0 ≤ s < m + 1.

Hence we have

$$\forall \epsilon > \epsilon_0, \exists K \in N\ s.t. \forall p,q \ge K \Rightarrow \left\| x_p - x_q \right\|$$
$$\le \left\| x_p - v_{r+1} \right\| + \| v_{r+1} - v_{s+1} \| + \left\| v_{s+1} - x_q \right\|$$
$$= \left\| v_{(r+1)_p} - v_{r+1} \right\| + \| v_{r+1} - v_{s+1} \| + \left\| v_{s+1} - v_{(s+1)_q} \right\|$$
$$< \frac{\epsilon - \epsilon_0}{2} + \epsilon_0 + \frac{\epsilon - \epsilon_0}{2} = \epsilon.$$

Thus {xp} is an ε_0-Cauchy sequence in D. And it is obvious that SSL({xp}) = V . Hence we have

$$\boxed{\varepsilon_0 - lim}_{p\to\infty} x_p = \bigcap_{\alpha \in V} \overline{B}(\alpha, \varepsilon_0) = \overline{T_m}(V, \varepsilon_0).$$

Since $D \cap \overline{T_m}(V, \varepsilon_0) = \emptyset$ by assumption, this implies that {xp} is not an ε_0-convergent sequence in D. ThusD is not ε_0-complete. This contradiction completes the proof. □

Proposition 2.15. Let $\varepsilon_0 > 0$ be a positive real number. If a subsetD of Rm is not ε_0-complete then there is an ε_0-Cauchy sequence {xp} such that $SSL \cap B(\gamma, r) = \emptyset$, $diam(SSL) = \varepsilon_0$ for some vector γ ∈Rm and some positive real number r >0. Moreover, SSL satisfies the following condition.

∀α ∈SSL,∃β ∈SSL({xp}) s.t. $\|\alpha - \beta\| = \varepsilon_0$.

Proof. Suppose that D is not ε_0-complete. Then there exists an ε_0-Cauchy sequence {xp} in D such that D∩$\boxed{\varepsilon_0 - lim}_{p\to\infty} x_p$ = ∅. If SSL∩D ≠∅ then we have

∅≠D ∩ SSL ⊆D ∩ { $\bigcap_{\alpha \in SSL} \overline{B}(\alpha, \varepsilon_0)$ } ⊆D ∩ $\boxed{\varepsilon_0 - lim}_{p\to\infty} x_p$.

This is a contradiction. Since SSL({xp}) ⊆D, this contradiction implies that SSL({xp}) ⊆ $D' - D$. But there are elements γ ∈Rm and r >0 by proposition 2.7 such that hull(SSL) ∩ B(γ,r) $\neq\emptyset$ and $\overline{B}(\gamma,r)\subseteq\bigcap_{\alpha\in SSL}\overline{B}(\alpha,\varepsilon_0)$.

And if SSL ∩ B(γ,r) $\neq\emptyset$ then there is an element α0 ∈SSL $\subseteq D'-D$ such that α0 ∈B(γ,r). Since α_0 is an accumulation point of D and B(γ,r) is an open set, there is an element x ∈D such that x ∈B(γ,r).Hence we have $D\cap\{\bigcap_{\alpha\in SSL}\overline{B}(\alpha,\varepsilon_0)\}\neq\emptyset$ which is a contradiction. Thus we have SSL∩B(γ,r) = ∅. Moreover, suppose that there is an element α0 ∈SSL({xp}) such that $\|\alpha_0-\beta\|<\varepsilon_0$ for all β ∈SSL({xp}). Thenwe have

$$\max\{\|\alpha_0-\beta\|:\beta\in SSL(\{x_p\})\}=r_0<\epsilon_0$$

since the set SSL({xp}) is compact. Then we have

$$\alpha_0\in B(\alpha_0,\epsilon_0-r_0)\subseteq\bigcap_{\alpha\in SSL}\overline{B}(\alpha,\epsilon_0).$$

Since α0 $\in D'-D$ and $B(\alpha_0,\varepsilon_0-r_0)$ is an open set containing α0, we have $D\cap B(\alpha_0,\varepsilon_0-r_0)\neq\emptyset$. This is a contradiction as the above. Thus we have

∀α ∈SSL,∃β ∈SSL({xp}) s.t. $\|\alpha-\beta\|=\varepsilon_0$.

Since the diameter of SSL is not greater than ε_0, this implies that $\mathrm{diam}(SSL)=\varepsilon_0$. □

Theorem 2.16Let $\varepsilon_0>0$ be a positive real number and D be a subset of Rm. Then D is not ε_0-complete if and only if there is a compact subset S of $D'-D$ such that $diam(S)=\varepsilon_0$ and $D\cap\{\bigcap_{\alpha\in S}\overline{B}(\alpha,\varepsilon_0)\}$.

Proof. ($\Rightarrow$) Suppose that D is not ε_0-complete. Then there is an ε_0-Cauchy sequence {xp} such that D∩ $\boxed{\varepsilon_0-lim}_{p\to\infty}x_p$ = ∅. By the propositionjust above, we have SSL({xp}) ⊆ $D'-D$ and $diam(SSL(\{x_p\}))=\varepsilon_0$.Now put S = SSL({xp}). Then S is compact by lemma 1.3. And $diam(S)=\varepsilon_0$ and S $\subseteq D'-D$ by the proposition just above. Moreover,

$$D\bigcap_{\alpha\in S}\overline{B}(\alpha,\epsilon_0)=D\cap\{\bigcap_{\alpha\in SSL}\overline{B}(\alpha,\epsilon_0)\}=\emptyset$$

since $\bigcap_{\alpha\in SSL(\{x_p\})}\overline{B}(\alpha,\varepsilon_0)=\boxed{\varepsilon_0-lim}_{p\to\infty}x_p$. (⇐) Suppose that there is a compact subset S of $D'-D$ such that $D\cap\{\bigcap_{\alpha\in S}\overline{B}(\alpha,\varepsilon_0)\}=\emptyset$ and $diam(S)=\varepsilon_0$. We can write as S = {sj:j ∈J} for some index set J. Since S $\subseteq D'-D$, for each j ∈J, there is a single-valued sequence $\{x_{j_p}\}$ in D such that $\|x_{j_p}-s_j\|<\frac{1}{p}$ for each p∈N. In order to prove that D is not ε_0-complete, let's choose a multi-valued sequence {xp} so that xp= {x_{j_p}: j ∈J} for each p ∈N. In order to show that {xp} is an ε_0-Cauchy sequence, let any positive real number $\varepsilon>\varepsilon_0$ be given. Choosing a natural number K ∈N so large that $K>\frac{2}{\varepsilon-\varepsilon_0}$, we have, since $\|s_j-s_k\|\leq\varepsilon_0$ for all j,k∈J,

$$\forall\varepsilon>\varepsilon_0,\exists K\in N\ s.t.(\forall p,q)p,q\geq K,\forall x_{j_p}\in x_p,\forall x_{j_q}\in x_q$$

$$\Rightarrow\left\|x_{j_p}-x_{k_q}\right\|\leq\left\|x_{j_p}-s_j\right\|+\left\|s_j-s_k\right\|+\left\|s_k-x_{k_q}\right\|$$

$$\leq\frac{1}{p}+\epsilon_0+\frac{1}{q}\leq\frac{2}{K}+\epsilon_0<\epsilon-\epsilon_0+\epsilon_0=\epsilon.$$

Therefore, the sequence {xp} is an ε_0-Cauchy sequence in D. Since the limit of the sub-sequential limits is also a sub-sequential limit, we haveSSL({xp}) = $\overline{S}$. But $\overline{S}$= S since S is closed. Thus SSL({xp}) = S.Finally, we have

$$D\cap\{\bigcap_{\alpha\in SSL(\{x_p\})}\overline{B}(\alpha,\epsilon_0)\}=D\cap\{\bigcap_{\alpha\in S}\overline{B}(\alpha,\epsilon_0)\}=\emptyset$$

by the assumption. Consequently, D is not 0-complete. □

Definition 2.17. Let D be a subset of Rm and f :D → Rn be a multi-valued function. We define thatf is ε_0-uniformly continuous onD if and only if we have

$$\forall\varepsilon>\varepsilon_0,\exists\delta>0\ s.t.(\forall x,y\in D)\|x-y\|<\delta,\forall f(x),\forall f(y)\Rightarrow\|f(x)-f(y)\|<\varepsilon.$$

Proposition 2.18. (Criterion) Let f :D → Rn be a multi-valued function defined on a bounded subset D of Rm. Then f is ε_0-uniformly continuous on D if and only if {f(xp)} is an ε_0-Cauchy sequence in Rn for every 0-Cauchy sequence {xp} on D.

Proof. ($\Rightarrow$) Suppose thatf is ε_0-uniformly continuous on D and any0-Cauchy sequence {xn} on D be given. Then we have

$$\forall\varepsilon > \varepsilon_0, \exists\delta > 0 \ s.t. (\forall x, y \in D) \|x - y\| < \delta, \forall f(x), \forall f(y) \Rightarrow \| f(x) - f(y) \| < \varepsilon.$$

Since {xn} is a 0-Cauchy sequence, we have

$\exists K \in N$,s.t.$(\forall p,q \in N)p,q \geq K, \forall x(p), \forall x(q) \Rightarrow \|x(p) - x(q)\| < \delta$.

Hence we have

$$\forall\varepsilon > \varepsilon_0, \exists K \in N \ s.t. (\forall p, q \in N) p, q \geq K, \forall f(x_p), \forall f(x_q) \Rightarrow \| f(x_p) - f(x_q) \| < \varepsilon.$$

Thus {f(xp)} is an ε_0-Cauchy sequence in Rn. ($\Leftarrow$) Suppose that f is not ε_0-uniformly continuous on D. Then we have

$\exists\varepsilon_1 > \varepsilon_0$ s.t. $\{\forall\delta > 0, \exists x\delta, y\delta \in D, \exists f(x\delta), f(y\delta) \in Rn \ s.t. \|x_\delta - y_\delta\| < \delta, \|f(x_\delta) - f(y_\delta)\| \geq \epsilon_1\}$.

Choosing $\delta = \frac{1}{p}$ for each natural number p $\in$N, we have

$\exists\{xp\},\{yp\} \subseteq D \quad \wedge \exists\{f(xp)\},\{f(yp)\} \subseteq$Rnsuch that $\|x_p - y_p\| < \frac{1}{p}$ *and* $\|f(x_\delta) - f(y_\delta)\| \geq \epsilon_1$.

Since {xp} and {yp} are bounded sequences in a bounded subset D and the closure D is compact, we may assume that $\lim_{p\to\infty} x_p = \lim_{p\to\infty} y_p = \alpha$ forsome α ∈D by choosing the single-valued and convergent subsequences. Now define a sequence {zp} by z2p−1 = xpand z2p = ypfor each natural number p ∈N. Then $\lim_{p\to\infty} z_p = \alpha$ and {zp} is a 0-Cauchy sequence in D. But we have

$$\|f(z_{2p-1}) - f(z_{2p})\| = \|f(x_p) - f(y_p)\| \geq \epsilon_1$$

for all p ∈N. Hence {f(zp)} is not an ε_0-Cauchy sequence. This contradiction implies the ε_0-uniform continuity of f on D. □

Theorem 2.19. Let f :D→ Rn be a multi-valued function defined on a 0- complete subset D of Rm. If f is ε_0-uniformly continuous on D then, for every 0-Cauchy sequence {xp} on D, there is an element α ∈D such that $\{f(x_p)\}$ is ε_0-convergent to f(α) ∈f(D).

Proof. Let any 0-Cauchy sequence {xp} on D be given. Since f(x) is ε_0-uniformly continuous on D, we have

$\forall\varepsilon > \varepsilon_0, \exists\delta > 0$ s.t. $(\forall x,y \in D)\|x - y\| < \delta, \forall f(x), \forall f(y) \Rightarrow \|f(x) - f(y)\| < \epsilon.$

But we have 0 − limxp= {α} for some α ∈D since D is 0-complete.Hence we have

$\exists K \in N$ s.t.$\forall p \geq K, \forall xp \Rightarrow \|xp - \alpha\| < \delta$.

Hence we have

$\forall\varepsilon > \varepsilon_0, \exists K \in N$ s.t.$\forall p \geq K, \forall f(xp), \forall f(\alpha) \Rightarrow \|f(x_p) - f(\alpha)\| < \varepsilon$.

Thus we have f(α) ∈ $\boxed{\varepsilon_0 - lim}_{p\to\infty} f(x_p)$ for all values of f(α). Since f(α) ∈f(D) for all values of f(α), the sequence {f(xp)} is an ε_0-convergent sequence of f(D). □

III. REFERENCE

[1] R. G. Bartle, The Elements of Real Analysis, John Willey and Sons (1964).
[2] D. L. Cohn, Measure Theory, Berkhauser Boston. (1980).
[3] B. H. Kang, An Introduction to ε_0-Density and ε_0-Dense Ace, JCMS vol31.(2018.2)
[4] H. L. Royden, Real Analysis, Macmillan pub. co. (1968).
[5] W. Rudin, Principles of Mathematical Analysis, McGraw Hill (1964).