Special Functions and the Generalized Integral

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Abstract. The paper introduces special functions that behave radically different from traditional functions. They are 'functions' that are not really functions in the traditional sense because they are either set-valued or their derivatives are. Some of them are called mischievous or pathological cases because they are counterexamples to well-established theorems. However, once some remedy is found that tame their strange behavior they become foundations of new and advanced mathematics. We focus on three of them: set-valued function, set-valued derivative and wild oscillation. With wild oscillation as the counterpart of "civilized" function of Schwartz distribution it is used in this paper to develop the generalized integral. The generalized integral is applied to quantum gravity to calculate the energy of a photon.
Keywords. Complementarity of speed and existence, Expectation, Generalized derivative, Generalized limit, Heisenberg uncertainty principle, Planck's constant, Probability distribution, Qualitative analysis, Wild oscillation.

I. INTRODUCTION

We introduce here 'functions' that are not really functions in the traditional sense because their derivatives are set-valued or the functions themselves are set-valued or both. Some of them are counterexamples to well-known theorems.They are called pathological cases. However, when remedies are found, they serve as foundations of new and advanced mathematics useful for science and engineering.

II. SPECIAL FUNCTIONS

References [1,2] focus on the critique-rectification of the real and complex number systems and their foundations as well as their extensions. The rectification of the real number system is the constructivist real number system [2] which contains the latter as its countably infinite subspace. We look at functions that reveal problems with the way we currently deal with functions, particularly, functions that serve as counterexamples to properties of traditional functions. Then we turn around and craft them into new tools for mathematics and science.

2.1 The Infinitesimal Zigzag

Consider triangle ADB of Figure 1. We define a sequence of polygonal curves C1, C2, …, Cn, …, as follows: Start with the curve C1,

(1)C1: $y1 = y1(x), 0 \leq x \leq 1$;

where y1(x) = the ordinate of the point above x on side AD or DB. Let P, Q and R, be the midpoints of the segments AD, AB and DR, respectively. Then PD || QR and PQ || DR so that $|PD| = |QR|$ and $|PQ| = |DR|$, where $|PD|$ = length of PD, $|QR|$ = length QR, $|PQ|$ = length of PQ and $|DR|$ = length of DR. The second term in the sequence of functions is C2: $y2 = y2(x), 0 \leq x \leq 1$, where y2(x) is the point in the polygonal line APQRB above x. We do similar construction on the polygonal line APQRB to define the third curve C3 and continue this scheme to generate the sequence of curves.

(2)C1: $y1 = y1(x), 0 \leq x \leq 1$,
C2: $y2 = y2(x), 0 \leq x \leq 1$,
C3: $y3 = y3(x), 0 \leq x \leq 1$,
……………………….
Cn: $yn = yn(x), 0 \leq x \leq 1, n = 1, 2, …$,

obtained by continuing similar construction on every preceding polygonal line. We replicate the construction on the succeeding polygonal lines to generate the sequence C1, C2, C3, …, Cn, …

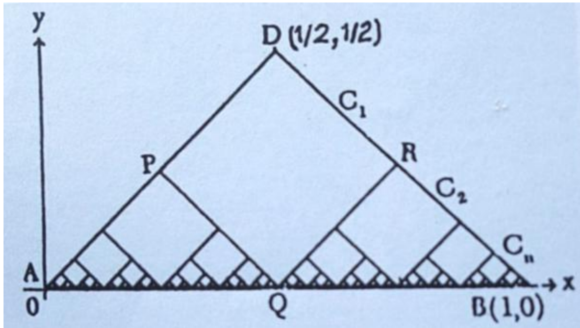

Figure 1. The first two terms C1 and C2 of the sequence of polygonal lines that appears to tend to the line y(x) = 0 in the sup norm and pointwise but it does not. On the contrary it tends to the infinitesimal zigzag C0: y0 = 0, $0 \leq x \leq 1$, as $n \to \infty$, in the interval [0,1]. (Figure from [3])§

Let $|Cn|$ = length of $|Cn|$, for each n, n = 1, 2, …, Cn. From Figure 1,

(3) $|C1| = \sqrt{2} = |C2| = |C3| = \ldots = |Cn-1| = |Cn| = \sqrt{2}.$

Note that lim sup $|Cn| = \sqrt{2}$, as $n \to \infty$ but the ordinary curve C: y(x) = 0, $x \in [0, 1]$, has $|C| = 0$. The curve lim Cn, as $n \to \infty$ (where we identify a function with its image, a curve) is called infinitesimal zigzag [4,5] denoted by lim Cn = C0: y0 = 0, $0 \leq x \leq 1$, as $n \to \infty$.The infinitesimal zigzag is distinct from the line segment C: y(x) = 0,$x \in [0, 1]$ because $|C| = 1$ but $|C0| = \sqrt{2}$. Note further that from the geometry of Figure 1, lim sup $|Cn|$ = lim $|Cn|$ pointwise which is$\sqrt{2}$.

Moreover, the sequence Cn: yn = yn(x), $0 \leq x \leq 1$; n = 1, 2, ..., of (1) is uniformly convergent point-wise and in the sup norm since each Cn is continuous and lim Cn = C0: y0 = 0, $0 \leq x \leq 1$, as $n \to \infty$, which is continuous. In fact, y0 coincides with y(x) = 0, $x \in [0, 1]$, which is absolutely continuous. Hence, y0 is also absolutely continuous.

Does y0′ exist in the interval [0,1], where y0′ = limsup yn′(x), $x \in [0, 1]$, as $n \to \infty$)? The answer is no since yn′ does not converge to a single value but keeps wobbling along the sequence +1, −1, +1, …, which has two limit points, +1 and −1, in the sup norm and point-wise. Therefore, it has set valued derivative. This result tells us several things about a function.

(a) Inadequacy of the present concept function; this was pointed out in [4,5] 80 years ago and, again, more recently in [6]. A function defined by its values alone cannot distinguish the function C: y = 0 from C0:y0 = 0 which are distinct in at least two ways: one is differentiable and the other is not and they also have different lengths.

(b) A function f(x) is more adequately represented as an ordered pair (f(x), f′(x)), where f′(x) is the derivative of f(x). What if the derivative does not exist or set-valued? Then the next observation applies.

(c) Inadequacy of the concept derivative; that the derivative of a function cannot be adequately expressed by the values of a function because derivative is a property belonging to an extension of its underlying space (extension of n-space to (n+1)-space in the general case) whose restriction to the space of real-valued functions contradicts some of its properties (e.g., property of absolute continuity). Therefore, there is a need to extend the conceptfunction to include those with set-valued derivatives. Also, the present defect in the conceptlimit is passed on to other concepts defined by limits including the derivative [4,5,6].

(d) In traditional mathematics, the derivative of a function is derived from the function so that it is dependent on the function. This is a limitation on the concept "function" because it rules out functions having no derivative in the traditional sense. Therefore, to broaden the space of functions, the derivative must belong to an independent space so that a function in n-space is more adequately represented by the pair (f(x), g(x)), where g(x) belongs to an independent space. This is particularly useful in control theory [6].

(e)Thus, the special function, C0: y0 = 0, $0 \leq x \leq 1$, is distinguished from the ordinary function y = 0 or the line segment AB. To be precise, we represent the infinitesimal zigzag by C0:(y0,y0'): y0 = 0, y0' = ± 1, $0 \leq x \leq 1$, which is a special function and a counterexample to a theorem in [7] that says,

An absolutely continuous function is differentiable, almost everywhere.

C0 is absolutely continuous but nowhere differentiable.

The infinitesimal zigzag, C0:(y0,y0'): y0 = 0, y0' = ± 1, $0 \leq x \leq 1$, belongs to a wider class of curves called generalized curves [5] different from the ordinary curve C: y = 0, $0 \leq x \leq 1$. Yet their values coincide point-wise. Furthermore, their arc lengths differ; in fact, there are countably infinite functions of this kind. One can see that although the sequence of functions Cn, n = 1, 2, …, converges to the segment AB point-wise or in the sup norm, its standard limit is something else: the infinitesimal zigzag, $C_0: y_0 = 0$, $y_0' = \pm 1$. This example raises two very important points:

(1) Fallacious proof of existence of a mathematical object by approximation or convergence as well as the erroneous use of numerical and algorithmic methods without existence theory (in fact, this flaw is a variant of vacuous statement [1]). This means that convergence in the norm does not suffice to show the existence of the limit of a function. For example, $\lim 1/x \neq 0$, as $x \to \infty$. Rather, $\lim 1/x = d^* > 0$, as $x \to \infty$, where d^* is the dark number [2].

(2) The inadequacy of the values of a function in characterizing its derivative; thus, the present notion of derivative is inadequate to capture the complexity of the properties of a function.

Equation (3) tells us that even if the set at which the function's derivative has measure 0, it cannot be ignored, especially, in integration.

For our purposes, we shift the notation a bit and denote the sequence of polygonal lines by $K_1, K_2, \ldots, K_n, \ldots$ If we take any continuous curve in the plane we can deform it into a sinusoidal curve of a given length and superpose a sequence of such curves over K_n, $n = 1, 2, \ldots, n, \ldots$, and call the sequence of such superposed curves, C_n, $n = 1, 2, \ldots, n, \ldots$, with $|C_n| = |K_n| = \sqrt{2}$ for each $n = 1, 2, \ldots$ (Figure 2).

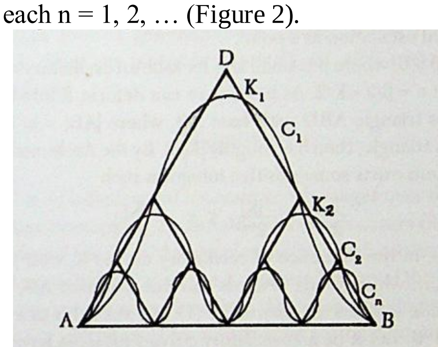

Figure 2. The sequence of sinusoidal curves, C_n, $n = 1, 2, \ldots$, superposed on the sequence of polygonal lines, K_n, $n = 1, 2, \ldots$, where $|C_n| = |K_n|$ for each n, $n = 1, 2, \ldots$ (Figure taken from [3])§

Taking the limits of both sequences, we have $\lim |C_n| = \lim |K_n| = \sqrt{2}$, as $n \to \infty$. Thus, there is an infinity of curves whose lengths are all equal to $\sqrt{2}$ that coincide with but are distinct from C: $y(x) = 0$, $0 \leq x \leq 1$. The result can be generalized for curves of any length > 0 using the same construction.

We feature the sinusoidal curve because it is a universal configuration of matter. For example, the profile of water wave is sinusoidal. So is the profile of electromagnetic wave. The superstring, fundamental building block of matter, is a circular spiral helical loop [8] and the projection of the helix on a plane through its axis is sinusoidal [9]. In its non-agitated state (dark) its cycle length is less than 10^{-16} m and not observable by visible light [10]. However, when suitably agitated by electromagnetic wave, it converts into a primum, unit of visible matter [10]. The basic prima are the electron, charge –1; +quark, charge +2/3; and –quark, charge –1/3 [11]. They are basic because they comprise every atom [12]. The electron serves as connectors (valence electrons) between two atoms, one from each component atom, in the formation of a molecule [10].

2.2. Applications of the infinitesimal zigzag

Recall that the derivative of y_0 the sequence $y_n' = +1, -1, +1, -1 \ldots$, does not converge to a single point, as $n \to \infty$, since its set limit is $\{-1, +1\}$, i.e., y_0' is set-valued. This is true in the sup norm or the metric induced by point-wise convergence. The infinitesimal zigzag is an example of a generalized curve [5], i.e., a function with set valued derivative. The solution [5] of the calculus of variations problem or problem 23 of Hilbert's Problem [13] is a generalized.

Consider this problem: Find the minimum of the integral,

(1) $\int_{[0, 1]}((1 + x^2)(1 + ((x'^2 - 1)^2)^{100})dt$

(where x' is derivative) from 0 to 1 among admissible functions $x(t)$ subject to $x(0) = x(1) = 0$. In traditional mathematics the "obvious' optimal curve among conventional curves subject to $x(0) = x(1) = 0$ is $x = 0$, $x' = 0$, $0 \leq x \leq 1$, and the minimum is 2^{100}. However, by admitting infinitesimal zigzag, which is like the ordinary curve $x = 0$ but whose derivative is set-valued and concurrently takes the values +1 and −1, and attaching a probability weight 1/2 (unit measure distribution of the weighted average when finite valued) to each of these values, we obtain a minimum of 1. Thus, conventional curves yield incorrect solution of this variational problem. The integral (4) is called Young measure [6], the natural norm for the calculus of variations.

As remedy for the anomalous behavior of the curve C0, we put into account the behavior of its derivative by representing a function parametrically as a pair, C: (f(t), g(t)), $t \in [0, 1]$, where g is the derivative of f. Then the natural metric for purposes of optimization is the Young measure, which is a curvilinear integral of the function f as the objective or cost function [6] in the interval [0, 1]. If we represent that measure by the integral,

(2)I(C) = ∫[0, 1](f(t),g(t))dt,

then I(C) is the Young measure of the curve C. When the integrand is 1, I(C) is called the length of the curve. Thus, a curve is a linear functional and curves of the same Young measure belong to the same equivalence class representing that linear functional. This makes functional analysis available to optimal control theory. In an optimal control problem the derivative g is the control parameter so that it is independent of f; in other words, the system is controlled by a finite set of values of the derivative.

We make references to the superstring,fundamental building block of matter [8]. Its structure is called nested generalized physical fractal sequence of superstrings [13]where the first term is a close helix; it has a flux called toroidal flux, a superstring, in its helical cycles traveling at 7 × 1022 cm/sec or 1012 times the speed of light [14]; the toroidal flux has a toroidal flux, a superstring, in its helical cycles traveling at the same speed, etc. (We shall discuss later this repetitive structure called fractal that makes the superstring indestructible)

Given any curve in the plane we can deform it into an oscillatory curve y = simmbx which is rectifiable; we can further deform it into some isosceles triangle ADBso that its length is preserved and equal to the sum of the lengths of AD and DB. In turn, we can deform this triangle into a finer oscillatory curve K1, with length preserved (Figure 2). We iterate this deformation forming an alternate sequence of polygonal lines and oscillatory curves Kn from A to B. Again, the sequence Kn tends towards a generalized curve called infinitesimal oscillation whose function component coincides with the zero function C: $y= 0, 0 \le x \le 1$. Its length is equal to the original length $|K|$ of K and its derivative at any point $x \in [0,1]$ is set-valued and equals the set of limit points of the derivatives of the sequence of oscillations at x. Since the segment AB is arbitrary we can prescribe its length to be an arbitrary number $\varepsilon > 0$. Then we have the following:

Theorem 1. Given an oscillatory curve K, any number $\varepsilon > 0$ and a line segment AB, there exists a continuous deformation of K into a fine oscillatory curve inside an ε-neighborhood of AB that preserves the length of K [15].

Proof. The proof uses the same construction as in Figure 1 with the triangle adjusted so that length of K = $|AD| + |DB|$.

Theorem 2. Given an oscillatory curve K, there exists a continuous deformation of K, with length preserved, into an arbitrarily small neighborhood of a point [15].

Proof. We prove both theorems. Let A be a given point and B a point in the ε-neighborhood of A and suppose $|AB| = \varepsilon/2 > 0$. There exists a deformation of K, with length preserved, into two sides of an isosceles triangle ADB where $|AD| + |DB| = |K|$. Following the construction above there exists a sequence of polygonal curves Cn and corresponding oscillatory curves Knsuch that for each n, $|Kn| = |Cn| = |AD| + |DB| = |K|$ and Kn tends to the segment AB (Figure 2). Hence there exists a positive integer N such that whenever $n \ge N$, the curve Kn lies inside the ε-neighborhood of A. (This establishes the first theorem) since the length of AB is arbitrary, $\varepsilon > 0$ and $|AB| = \varepsilon/2$. Then the second theorem follows from the first. □

Note that in Figure 2 the oscillatory structure is preserved as well as its length. Thus, it is possible to shrink an oscillatory curve of any length into an infinitesimal oscillation at a point. Now, let $\beta > 0$, where β is small, and let K be an oscillatory curve of large length K. Let $\varepsilon = \beta/2 < |K|/2$. As before, we deform K into the two sides of an isosceles triangle ADB with base AB, where $|AB| = \varepsilon$. Let h be the altitude of this triangle, then for suitably small ε , $h \approx |K|/2$. By the Archimedean property of the decimals [2] there exists some positive integer n such that,

(3) $|K|/2n+2< |K|/2n+1< |K|/2n$

Therefore, in the sequence of oscillatory curves Ki with $|Ki| = |K|$, for each i = 1, 2, ..., which tends towards the line segment AB, there is one whose amplitude satisfies the inequality (3). We state this as a theorem.

Theorem 3. Let K be an oscillatory curve with large length $|K|$ and let $\varepsilon > 0$, $\varepsilon = \beta/2 < |K|/2$. Then one can continuously deform the oscillatory curve K into an arbitrarily small neighborhood of a point with its length and amplitude prescribed to satisfy,

(4) $|K|/2n+2 < |K|/2n+1 \le \varepsilon \le |K|/2n$,

for some integer n [15].

Chaos is mixture of order none of which is identifiable [16]. The following theorem is now obvious and follows from the above theorems:

Theorem 4. The real line is chaos.

Theorems 1 – 4 model different aspects of the shrinking of a superstring. They have other implications for physics that can explain certain phenomena such as the tremendous but undetected (latent) energy in the nucleus of an atom. Tremendous because we can pack infinitesimal helical loops (e.g., the superstrings) into an arbitrarily small neighborhood of a point at very high energy level $h\zeta$ where h is Planck's constant and ζ is the number of helical cycles.

We have already admitted a function with set-valued derivative and used the latter as part of the characterization of the former. This way, we enrich the admissible spaces of functions. In fact, this method of enrichment was introduced by L. C. Young in a series of papers involving construction of complete spaces where the calculus of variations problem has a solution [4,5,17,18]. The solutions in these cases are generalized curves and surfaces. The idea is to represent a function parametrically as an ordered pair (f(t), g(t)), where g(t) is the derivative of f(t). In the case of finding a curve that minimizes the curvilinear integral (4), the solution is a generalized curve, i.e., the limit of a sequence of piece-wise constant curve in the Young measure. In our example the limit of the sequence of piece-wise constant curve is the infinitesimal zigzag.

III. SET-VALUED FUNCTIONS

We go further beyond Young by admitting set-valued functions, not just derivatives. We consider functions of the form,

(1)Sinm1/x, (sinn1/x)(cosxm1/x),

where m and n are integers. This is set-valued at the origin.

3.1 The wild oscillation sin1/x

The wild oscillation F(x) = sin1/x is a special case of the more general wild oscillation sinm1/xk, where k, m, are positive integers. It reveals a flaw in the Lebesgue theorem on the Riemann integral that says:

A bounded function is Riemann integrable if and only if its set of discontinuity has measure zero [7].

The bounded function F(x) = sin1/x whose only discontinuity is at x = 0 is not Riemann integrable in any neighborhood of the origin. Known proof of integrability of sin1/x involves construction of a Riemann integral outside an ε-neighborhood of x = 0, where $\varepsilon > 0$, which exists, and taking a sequence of such integral as $\varepsilon \to 0$, which converges. The limit of such a sequence, however, is not necessarily Riemann integrable, certainly, not sin1/x because no Riemann sum of this function can be formed in any neighborhood of 0. This is, in fact, a form of the Perron paradox [19] on the use of necessary conditions without proof of existence of a function with the given property. It is quite common in solving a differential: let f(x) be the solution; then f(x) is substituted in the given differential equation and if it satisfies the equation it is taken as the solution. It need not be the solution. The same fallacy applies to approximation of some object the existence of which is not known.

In the development of the Henstock integral in [20] the function sin1/x2 plays a central role. However, the theory is flawed by the inadequate conceptderivative. While this function is shrunk to zero by the factor x2,its derivative is not since it belongs to a space independent of the function. The function considered in [20] is F(x) = x2sin1/x2, $0 \le x \le 1$ (Figure 3). It is asserted that its derivative F′(x) exists at x = 0 and F′(x) = 0 because at that point its one-sided derivative can be trivially computed since, using the ordinary definition of derivative, we have,

(2) $|\Delta F| / |\Delta x| \le |x2| / |x| = |x|$,

so that lim $|\Delta F| / |\Delta x| = 0$, as $x \to 0+$, exists.

The inequality follows from the fact that F(x) is bounded by its envelope, the pair of parabolas y = x2 y = –x2. F(x) is continuously differentiable outside x = 0. In fact, we have, at $x \neq 0$,

(3)F'(x) = 2xsin1/x2 – (2/x)cos1/x2,

and its graph is shown in Figure 4, where the term 2xsin1/x2 has been discarded since it vanishes as $x \to 0+$. However, the term (2/x)cos1/x2 oscillates rapidly along all values in the interval $(-\infty,\infty)$ as $x \to 0+$ and does not converge.

This is a particular kind of discontinuity, an example of what we shall call chaos. Moreover, this is another example of the derivative of a function that is independent of it.

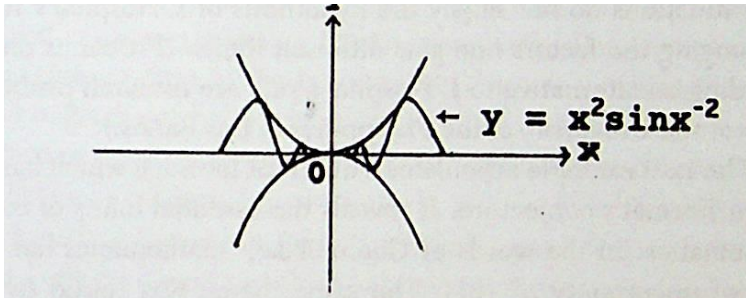

Figure 3. The function F(x) = x2 sin–2 has its envelope the pair of parabolas y = x2 and y = –x2 which are tangent to each other at ther vertices at the origin. (Figure from [3]) §

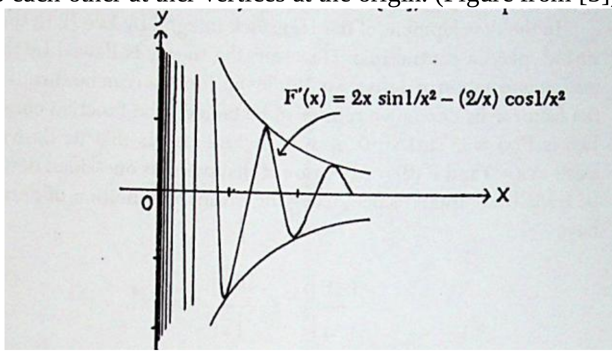

Figure 4.The graph of F′(x) = – (2/x)cos1/x2 where the term 2xsin1/x2 is discarded since it tends to 0 with x; it takes all values in (–∞,+∞) as x → 0. (Figure from [3]) §

Our final set-valued function is a function of the type,

(4)(e1/z/xk)(sinm1/x2 + cosn1/x2),

(e1/z xk)(sinn1/x2),

where z = x2, k, m, n are positive integers. Finding the limits of these functions, as x → 0, quickly reveals that L'Hospital's rule breaks down on (3). The reason: these functions do not satisfy its hypothesis at the origin, namely, that the function should not have a zero in any neighborhood there; each of the functions in (4) has countably infinite zeros in any neighborhood of the origin. Also by rearranging the factors one gets different standard limits. The generalized derivatives of (e1/z/xk)sinm1/x2 and (e1/z/xk)(sinm1/x2) or their expectations are evaluated in [3] and used to generalize L'Hospital's rule; the latter is applied to either of function (4) to evaluate their limits as x → 0 [3].

3.2 Rapid Helix and oscillation

A wild helix is the inverse projection of the function G(x) = (2/x)cos1/x2 into a cylinder around the x-axisjust outside an ε-neighborhood of the origin, where ε> 0 but a small number. Similarly, a rapid oscillation is a segment of the wild oscillation just outside an ε-neighborhood of the origin, where ε> 0 but small. A primum (unit of visible matter) is mathematically modelled by the rapid spiral x = t, r(t) = β(sinnπt)(cosm kπt), t ∈ [–1/k,1/k], θ =nt, n, m, k, integers, n >> k, m even, whose profile is a sinusoidal curve of even power [21]. Its cycle energy is Planck's constant h = 6.64×10–34joules [22], the irreducible unit of energy. Energy conservation and flux compatibility pull the primal cycles together to form a set-valued function that requires the generalized integral [3] to do calculation on it because the ambiguity or uncertainty of large number induces uncertainty on such large number of primal cycles.

3.3 'Tamer' Function

Instead of a nice function η suppose we take a wild oscillation {f(x)} and a "tamer" function that we denote by W(x) that "tames" {f(x)} in the sense that it provides structure to and approximates it. The structure we shall consider here is probability or unit measure distribution. The wild oscillation negates all the nice properties of η such as differentiability and being identically 0 at the ends. Not only is it not differentiable at 0 it is also set-valued there and its graph wobbles wildly between two values as x approaches 0. We tame its bad behavior at the origin by an integral of the form (1) with W the tamer function and the wild oscillation {f(x)} in place of η. We call such integral generalized integral, a dual of Schwartz distribution appropriate for wild oscillation in the sense that the wild oscillation negates the nice properties of the Schwartz distribution and the tamer affects it through the integral. An oscillation is wild if it tends to be set-valued at some point of its domain. We impose less restriction on f(x) by requiring only the existence of the second derivative. Thus, the tamer W(x) is not as smooth as η. However, it does something for a wild oscillation: it approximates and provides structure to and, therefore, tames it. The appropriate interval for our purposes is [0,π/2] but for the extended wild oscillation any interval [0,b] will do.

3.4 Examples of wild oscillations

(a) The wild oscillation, sinm1/x; it is set-valued at x = 0; we denote its set-value there by sinm/0 = slimsinm1/x, as $x \to 0+$ (slim means set limi; sinm/0).

(b) The wild oscillation sinm1(x – s), as $s+\to x$, s, $x \in [0,\pi]$; we denote its set-value at x by: sinm1/0x+ = slimsinm1/(x – s), as $s+\to x$ (at the right end of the interval we take, slimsinm1/(s – x), as $s\to x-$)).

(c) The product wild oscillation (sinm1/x)(cosm1/x), set-value: (sinm1/0)(cosm1/0) at x = 0.

(d) Slim(sinm1/(x – s))(cosm1/(x – s)); set-value: (sinm1/0x)(cosm1/0+)

= Slim(sinm1/(x – s))(cosm1/(x – s)), as $s+\to x$, s, $x \in [0,\pi]$.

There are obviously a number of variations of functions (a) – (d) but they will be our focus to illustrate the methodology and apply the generalized integral built on (a) to quantum gravity [23]. (For the different cases of wild oscillations and their generalized integrals see [3]) We call function (b) the extended function of (a) and function (d) that of (c).

IV. THE GENERALIZED INTEGRAL

We construct the generalized integral on the wild oscillations above using approximation by rapid oscillation [3]. A key principle in this paper is a mathematical model of the complementarity of existence and speed in physics of which the Heisenberg uncertainty principle [24] is a special case:

4.1 Oscillation probability principle.

The normalized derivative dq/dw of the approximating rapid oscillation g(x) = sin2x is the probability that the projection of the oscillating point P lies outside the subinterval [y,y+dy) in the set-value of the given wild oscillation.

The oscillation probability principle is an example of physical mathematics, i.e., mathematics derived from physical principle or based on a physical process. When a physical system or process is described by, say, a system of differential equations and the mathematical solution is known, a mathematical problem can sometimes be solved if it can be stated as a system of differential equations similar to or can be derived from the former. Physical mathematics has now become a tool in both physics and mathematics [25].

We shall use this principle to derive another key concept of generalized integration – probability distribution or unit measure distribution. A lot of times problems that are impossible or difficult to solve by computation alone are easily disposed of by qualitative or noncomputational analysis, i.e., pure reasoning based on mathematical or physical principles or rational thought [26]. We use it here liberally.

4.2 Probability distribution

A set without structure is uninteresting. For our purposes the appropriate structure on set-valued functions is probability or unit measure distribution. Measure distribution is any distribution of entities, e.g. density and pressure, over a line, surface or volume. Then in this example, the sum of density or pressure over a distance, an area or volume is force. It can be a variable. For instance, the water pressure along a vertical line is a function of depth. When the weighted average of these entities is divided by their total sum we call the quotient probability or unit measure distribution, i.e., normalized probability distribution. Since distribution is a sum we can integrate a function with respect to it and when the function varies over the range of a set-valued function such integral is called a generalized integral; it is particularly designed for integrating set-valued function with distribution, not necessarily probability distribution. For instance, it can be used for calculating the total force on a dam through generalized integration with respect to the pressure distribution.

We shall apply the generalized integral to quantum gravity. However, since probability distribution is simply normalized distribution of any kind such as pressure and density it has broad applications. It should be particularly useful when the distribution is not homogeneous. For example, blood pressure in the body is not only variable but dependent on a number of factors such as gravity, pumping of the heart and the position of the body (e.g., when lying down, sitting or standing) and when the blood vessels have blockages such as blockages due to stenosis.

4.3 The wild oscillation sinm1/x and its extended wild oscillation sinm1/0

Consider the wild oscillation, {f(x)} = sinm1/x where, for purposes of application, n is a large integer. Derivation of the probability distribution for this function is based on the oscillation probability principle using the rapid oscillation W(x) = sinmnx. We shall consider the more general case m an even integer since when m is odd computation is trivial in view of the symmetry of the probability distribution involved. The function {f(x)} is set-valued only at x = 0, its set-value being the vertical interval segment [0,1] denoted by sinm1/0, along the y-axis. It is the set of limit points of the projection of W(x) over a half-phase [0,π/2] corresponding to the unit interval [0, 1] on the y-axis. We put structure on it, its cross-sectional probability distribution along the vertical segment [0,1] at x = 0. We approximate it by the probability distribution of the rapid oscillation W(x) with m even and n a large integer. We note that as x → 0 the wild oscillation becomes more and more rapid and its arc on one period becomes more and more symmetrical with respect to the vertical through its maximum point and, therefore, is approximated more and more by the rapid oscillation W(x).

We note further that multiplying a function by a constant only alters its values, but not its relative values; it amounts to a change of scale. This is the basis of normalization of distribution to turn it into a probability distribution by dividing it by the sum of its values. Thus, for purposes of approximating probability distribution we do not even need the multiplier n; we use the ordinary function f(x) = sinmx over the interval [0,π/2] which corresponds to half a period of an arc above the x-axis since m is even. The effect of n is only to shrink its period so that the arc will approximate one period of an arc of the wild oscillation {f(x)}. Therefore, n is not necessary because it has no effect on the approximating probability distribution. In view of the symmetry of the rapid oscillation we need only a half arc that ranges over the interval [0,π/2]. Moreover, m only determines the bluntness at the maximum and flatness of base of the approximating arc and has insignificant effect on the distribution of values. Therefore, there is no loss of generality if we use the value m = 2.

Let x increase uniformly from 0 to π/2. Then the projection of the point P(x,w(x)) sweeps over the set-value [0,1] of {f(x)} at the origin. Divide [0,1] into the non-overlapping subintervals {[y,y+dy)}, as y ranges from 0 to 1 except the interval on the upper end of the segment which we take as [y,1], dy = 0 (we assume this exception from now on). We calculate the probability distribution in terms of the variation or distribution of derivative, i.e., relative variation of speed of the projection of P, over the half-arc in one sweep. Its speed on any of the subintervals is proportional to the derivative along corresponding subinterval of this half-arc. We drop the proportionality constant since that will be taken care off when we normalize the probability distribution. We ask: what is the probability that the projection of P lies outside the subinterval [y,y+dy)? By the oscillation probability principle, that probability is proportional to the speed of the projection of P, i.e., the derivative of dy/dw; again, we drop the probability constant. Of course, that probability is 0 outside [0, 1]

Denoting by dp/dw the probability that the projection of P on the interval lies in the subinterval [y,y+dy) and by dq/dw the probability that it lies outside the subinterval we have,

(1)dp/dw + dq/dw = 1 or dp/dw = 1 –dq/dw,

where w is a dummy variable for differentiation and, later, for integration. Since

(2)dq/dw= 2sinwcosw,

we have,

(3)dp/dw = (1 – 2sinwcosw)dw,

where dp/dw may not be normalized. To normalize (3) we first note that the projection of P lies in the interval [0,1]; therefore, we divide dp/dw by the integral,

(4)∫[0,π/2] (1 – 2sinwcosw)dw = (w − sin2w)| [0,π/2] = 1 − π/2 = (2 − π)/2.

We take the positive value of this normalizing constant; then the normalized probability distribution is given by

(5)dp/dw = 2(2sincosw − 1)/(π − 2)))dw.

We compute the expectation of the set value [0,1] of {f(x)} (and call it its generalized derivative (GD):

(6)GD({f(x)}) = E({f(x)}) = 2∫[0,π/2] ((2sinwcosw − 1)/(π − 2)))sin2wdw

= ∫[0,π/2] 2((2sin3wcosw − sin2w)/(π − 2))dw

= 2((1/2)sin4w) − (w/2 − (1/4)sin2w))/(π − 2))| [0,π/2]

= 2((−1/2) + (π/4))/(π − 2) = (2/4)(π − 2)/ (π − 2) = 1/2.

This is the approximate expectation of {f(x)} = sinm1/x. Note that the inflection point of the rapid oscillation at x = π/4, and f(π/4) = 1/2 distorts the actual distribution; it has the same effect on distribution as compact support has on a function: its values are concentrated in it. In this case its counterpart is the singleton {1/2}. Normally, without the inflection point, the expectation of both the approximating half-arc of f(x) and E({f(x)}) would have been near the base of the curve on the x-axis due to the flatness of both f(x) and an arc of {f(x)} near the origin but the inflection point where the projection of P stops momentarily, skews the probability distribution up and makes it coincide with the inflection point, i.e., GD({f(x}) = 1/2. Therefore, to avoid the distortion we calculate the sum of the approximated expectations in the subintervals [0,1/2] and [1/2,1], i.e.,

(7)E({f(x)}[0,1] = E[0,1/2]({f(x)}) + E[1/2,1]({f(x)}) = ∫[0,π/4] 2(2sincosw − 1)/(π − 2)))dw

+ ∫[π/4,1] 2(2sincosw − 1)/(π − 2)))dw

= 2((1/2)sin4w) − (w/2 − (1/4)sin2w))/(π − 2))| [0,π/4]
+ 2((1/2)sin4w) − (w/2 − (1/4)sin2w))/(π − 2))| [π/4,π/2]
= 2(−1/8 + 1/4 + π/8) − 1/4 + π/4)) /(π − 2)) = (3π/4 −1/4)/(π − 2) = 0.3, which is the actual approximated E({f(0)}.
Since the probability distribution of {f[x]} = sinm1(x – s), as x → s+, s ∈ [0,π] is uniform it is constant in the interval [0,π], i.e., y = E({f[x]}) = 0.3. Therefore, the approximate weighted area of the extended wild oscillation {f[x]} in the interval [0,π] is given by
(8)∫[0,π] ∫[0,π/4] 2sinmw ((2sinwcosw − 1))/ (π − 2))dwdx = ∫[0,π] (1/2)dx = 0.3π.
Consider the integral,
(9)F(x) = ∫[0,x] (2∫[π/4,π] (2sincosw − 1) sin2w/(π − 2))dwdx
+ 2∫[0,π/4] ∫[π/4,π/2] (sincosw − 1) sin2w/(π − 2))dwdx.
Then we define the derivative of the double integral as the inner integral and we have an analogue of the fundamental theorem of the calculus:
(10)(d/dx)F(x) = 2∫[π/4,π/2] (sincosw − 1) sin2w/(π − 2))dw.
Here, w is a dummy variable for differentiation.

4.4 The wild oscillations (sinm1/x)(cosm1/x) and (sinm1/0x)(cosm1/0x)
We now develop the scheme for finding the probability distribution of the product wild oscillation given by
(11){g(x)} = (sinm1/x)(cosm1/x),
its set value being (sinm1/0)(cosm1/0) at x = 0. Here, the symbol sinm1/0 is the projection of sinm1/x of a small subinterval of [0, 1] on the y-axis. We approximate the derivative by the appropriate rapid oscillation g(x) = sinmnπx)(cosmnπx. Again, without loss of generality, we let m = 2, n = 1 so that
(12)dq/dw = (d/dw)(sin2wcos2w) = 2(sinw xcos3w − coswsin3w)dw.
This product function is symmetric with respect to the vertical line at its maximum. To get the maximum, let Sinxcosx(cos2x − sin2x) = 0. Then, the solutions in the interval [0,π/2] are x = 0, x = π/2, x = π/4 so that the maximum is at the midpoint x = π/4 and the minima are at the two end points x = 0 and x = π/2.
Just to have a sense of how the product function looks like we note that f(x) increases from 0 to 1 in the interval [0,π/2] and g(x) decreases from 1 to 0 in the same interval. They intersect at x = π/4, f(π/4) = ($\sqrt{2}$)/2. To the left of the intersection, f(x) < g(x) < 1 and to the right g(x) < f(x) and at the intersection f(x) = g(x). It follows that, 0 ≤ f(x)g(x) < g(x) on the left and 0 ≤ f(x)g(x) < f(x) on the right. Therefore, the curve lies under both curves and its maximum is beneath their intersection so that it has no inflection point (see Figure 5)). Since f(x) and g(x) intersect at their inflection point their product has no inflection point. Moreover, since the product function has degree 4 in the sine and cosine functions which are both less than 1 its expectation must be very small, very close to the x-axis. Note that we obtain more information about the problem by qualitative analysis. (Incidentally, the product of an oscillation with any function is an oscillatio§n)

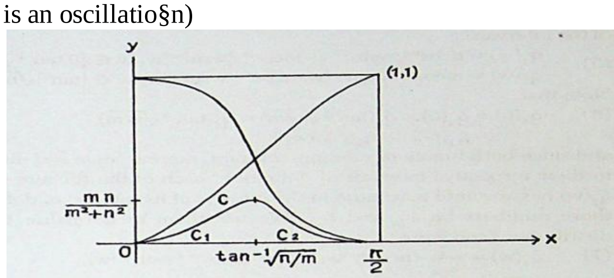

Figure 5.The oscillation curve C is the product of the oscillations y = sin21/x, y = cos21/x, each at half-phase [0,π/2]; the product curve C is at full face in [0, π/2].(Figure from [3])

Since, the sine and cosine functions are half a period shift from each other the product function is symmetric with respect to the vertical through its maximum. We, again, find the probability distribution on a half arc, i.e, the image of the interval [0,π/4] on the set value [0,($\sqrt{2}$/2)] along the y-axis under the product function.

We subdivide the vertical interval [0,$\sqrt{2}$/2] at the origin by the non-overlapping subintervals {[y,y+dy)}. Since this product function is symmetric the two branches of the arc have the same probability distribution and expectation so that it suffices to find the probability distribution in the interval [0,π/4].

We apply the oscillation probability principle again on the vertical interval [0,$\sqrt{2}$/2]. Since
(13)dq/dw = 2(sinw xcos3w − coswsin3w)dw,

then by the oscillation probability principle we have,
(14)dp/dw = 1 − 2(sinw xcos3w − coswsin3w)dw.
We find the normalizing constant,
(15)∫[0,π/4] (1 − 2(sinw xcos3w − coswsin3w)dw
= −π/4 − 2∫[0,π/4](sinw xcos3w)dw + 2∫[0,π/4]coswsin3w)dw
= −π/4 − 2(−1/4)cos4w |[0,π/4] + 2(1/4)sin4w |[0,π/4]
= −π/4 + 1/2 cos4w |[0,π/4] + 1/2 sin4w |[0,π/4] = −π/4 +2/8 = (1 − π)/4.
We take the positive value, (π − 1)/4 for the normalizing constant and reverse the terms of dq/dw to make it positive in the interval. Therefore, the normalized probability distribution is,
(16)dp/dw = 4(2(sinw xcos3w − coswsin3w) − 1)/(π − 1))dw.
To find the approximate expectation (E)or generalized derivative (GD) we evaluate,
(17)E({f(x)g(x)}) = 4∫[0,π/4] (2(sinwcos3w − coswsin3w) − 1)(sin2wcos2w)/(π − 2))dw
= 4∫[0,π/4] (2(sin3wcos5w − sin5wcos3w) − sin2wcos2w)/(π − 1))dw
= 4∫[0,π/4] (2(sin3w(1 – sin2w)2cosw − sin5w(1 − sin2w)cosw) /(π −1))dw
+ 4(sin2w(sin2w − 1)/(π − 1))dw
= 4∫[0,π/4] (2(sin3w(1 − 2sin2w + sin4w)cosw − (sin5w − sin7w)cosw/(π − 1))dw
+ 4∫[0,π/4] (sin4w − sin2w))/(π − 1))dw
= 4∫[0,π/4] 2(sin3w − 2sin5w + sin7w)cosw − (sin5w − sin7w)cosw/(π − 1))dw
+ 4∫[0,π/4] (sin4w − sin2w)/(π − 1))dw
= 4((1/2)sin4w − 2(1/6)sin6w + (1/8)sin8w − (1/6)sin6w/(π − 1)) |[0,π/4]
+ 4((1/8)sin8w)/(π − 1))| [0,π/4] + 4((−(w/2 + (1/4)sin2w)| [0,π/4]
− sin3wcosw/(π − 1))|[0,π/4 + 4(1/3)∫ [0,π/4] sin2w)/(π − 1)dw
= 4((−(1/8) + 1/24 − 1/128 + 1/48 − (1/28)/(π − 1) + (π/8 − (1/4)/(π − 1)
+ 4(1/4)(1/4) + (1/3)(w/2 −(1/4)sin2w/(π − 1))|[0,π/4] = 0.07.
Thus, the expectation or weighted average is very close to the base.

V. APPLICATION TO QUANTUM GRAVITY

This section is excerpted from [21]. Ito illustrates an application of the generalized integral to compute the energy of a photon modelled computationally as rapid oscillation. First we model basic primum computationally in cylindrical coordinates [21] by the helix, x = t, r(t) = β(sinnπt)(cosmπγx), θ =nπt, t ∈ [−2⁄πγ,2⁄γ], n, m, integers, n is odd, m is even, n >>γ and γ is a large positive real number depending on the length of the primum. (The primum any of the basic constituent of visible matter, namely, the electron +quark and −quark [10]). By the Energy Conservation natural law and quantization principle [8], its cycle energy is Planck's constant h = 6.64×10–34 J. Scooped up and carried by electromagnetic wave wave, its cycles flatten to rapid oscillation, x = t, y(t) = β(sinnπt)(cosmπγt) due to dark viscosity (dark matter consists of non-agitated superstrings [21])and becomes a photon (primum that has broken away from its loop and flattens into an oscillation in flight due to dark viscosity), y(t) = β(sinnπt)(cosmπγt). The energy of one full arc of a photon is h (one full cycle of the primum it comes from); its toroidal flux speed of 7 x 1022 cm/sec [15] is uniform along the arcs (or cycles in the case of the primum) regardless of length. Since the energy of a photon is known it is theoretically possible to find the number of cycles by dividing its energy by h (of course, energy varies the most energetic being violet and the least energetic red in the visible spectrum). Unfortunately, division by a small number less than 1 is inaccurate. (There is a limited number of digits of a decimal that the computer can compute accurately; beyond that accuracy collapses). The visible wave length of a photon is known (again, depending on its energy); therefore, uniformity of energy density allows its computation, say, in terms of Joules per cm; we denote it by γ. Then we can set up the generalized integral that computes the energy of the photon or the primum it comes from in terms of its probability distribution.
The computation uses both our earlier approximation of the wild oscillation by the rapid oscillation and its reverse, i.e., approximation of the latter by the former. We first express the probability distribution of the rapid oscillation as normalized energy density. The density is constant regardless of the length of the arc, by energy conservation. Since the envelope of the photon is symmetric with respect to its midpoint the total energy is four times the energy of the upper left half of its envelope y = ±βcos2πγt. There is no loss of generality if we take m = 2; the exponent only determines the shape and bluntness at the ends of the half arc but not significantly the energy content and distribution.
We compute the probability on a half arc of the rapid oscillation,
(1) y(t) = sinπnt,

Differentiating (1) we have,
(2) dq/dw = πncosπnwdw, dp/dw = (1 – πncosπnw)dw,
where w is the dummy variable for integration. To normalize dp/dw we note that the pre-image of the projection of the point P in the vertical interval at the origin lies in the interval [0,1/4n]. We divide the second equation of (2) by the integral,
(3) ∫[0,1/4n] (1 – πncosπnw)dw = (w − sinπnw ‖[0,1/4n]
= − 1/4n + sinπ/4 ≈ √2)/2,
since n is large. The normalized probability distribution is given by,
(4) dp/dw = √2(1 – πnsinπnw)dw).
Energy conservation requires that the distribution of energy be uniform among the arcs (or the cycles in the case of the primum) regardless of arc length or cycle length. Therefore, the energy density along the full length of the photon is also uniform. Let σ be the energy density in appropriate units along the photon's axis. We find the generalized integral in the vertical interval [0,1/4n] and the ordinary integral along the full length of the photon to find its total energy:
(5) (4σ√2)∫[0,1/4γ] ∫[0,1/4n] sinπnw(1 – πnsinπnw)cos2πγdwdx
= (4βσ√2)∫[0,1/4γ]∫[0,1/4n] (sinπnw –πnsin2πnw)cos2πγx dwdx
= (4βσ√2)∫[0,1/4γ] (−(1/πn)cosπnw − (w/2 − (1/4) sin2πnw)cos2πγx)dx ‖[0,1/4n]
= (4βσ√2)∫[0,1/4γ]((1/πn)cos(π/4) − (π/8 + (1/2)(√2/2)2) cos2πγx)dx
= (4βσ√2) ∫[0,1/4γ] (1/2πn + (π/8 − 1/4))dx ≈ 5.64(∫[0,1/4γ] 0.14dx = 0.79/γ J.
This can be checked with the known energy of the photon. From this value we can compute the numerical energy distribution of the photon. We can similarly compute the energy of a primum by considering the uniform energy density of its flattened projection to be concentrated on its envelope, calculating the sum along the full length of its profile and taking the full rotation of the latter suitably to find the total energy of the primum. Note that the total energy of the photon equals the total energy of the primum it comes from where the cycles convert to the arcs of the photon as rapid oscillation.
(We offer some scheme for avoiding the problem of dividing by the small number h. There are a number of ways. One is to temporarily invent a bigger unit of energy (change of scale) so that the order of magnitude of the divisor can be reduced considerably and another is to subdivide the interval of integration into suitable number of subintervals so that each will involve a small amount of energy or number of arcs or cycles. Addition does not amplify the margin of error in the result (division by small number does). With the generalized integral that is possible since each interval will determine a strip under the function. Or, we can have a combination of both)

VI. THE GENERALIZED LIMIT

This section is excerpted from the masteral thesis of the author's graduate student [27] (with slight editing by the author). It introduces the notion of generalized limit of set-valued function that leads to a generalization of L'Hospital's rule of elementary calculus.
In the ordinary sense, no general way of finding the limits of these functions exist:
(1)lim exp(−1/x2)[sinn(1/x2)(cosm1/x2)]/xk, as x → 0,
(2)lim exp(−1/x2)[sinn(1/x2) + (cosm1/x2)]/xk, as x → 0.
L'Hospital's rule for indeterminate forms of this type fails because of the existence of zeros in every neighborhood of the origin. To rectify this, we introduce a new concept of limit called generalized limit or Glim. Using this concept we may extend some theorems on limits to set-valued functions like the product theorem below. We first define the appropriate notion of limit of function that includes set limit.
Definition. Let f(x) be a function defined for all x in an open interval containing xo except possibly at xo. We define the set limit, Slim of f(x), as x → x0, as the set {f(x)} of projections of f(x) on the vertical line at x0 for all x in some ε-neighborhood of xo.
This means that the Slim of a function is actually the set of all its limit points. Moreover, this notion of limit reduces to the usual conceptlimit if the set {f(x)} is a singleton. Using this definition, we have the following, as x → 0,
(3)Slim sinn(1/x) = [−1,1] or [0,1],
depending on whether n is odd or even. We next define generalized limit, denoted by Glimf(x).
Note that the set limit of the product of two functions tends to 0 if one of the factors tends to and the other is bounded.

Definition. The generalized limit of the function f(x), as $x \to xo$, is the weighted average of its set limit.
This definition of generalized limit amounts to the expectation of the previous sections. To illustrate, consider the function,

$$(4) F(x) = \begin{cases} 1, & \text{if x is a terminating decimal)} \\ -1, & \text{if x is a nonterminating decimal,} \end{cases}$$

defined on the decimals which are discrete [2].
This function has two limit points, 1 and −1, as x approaches any number; thus, its set limit is {−1,1}. Since the rationals and and nonterminating decimals are both countable [2], the set of constructivist real numbers that tends to either limit point is countable (in the constructivist real number system uncountable number does not exist [2]). Therefore, the weighted average or expectation point is 0. Hence, Glim f(x) = 0, as $x \to xo$. We can see here the relationship between density and expectation point.
Since the dark number d* cannot be separated from any decimal [2] an interval or its image under a continuous function is a continuum. The concept of probability distribution is needed to determine the Glim since the weighted average of the set in the continuous case is no different from the expectation point of the set. Thus, for set-valued functions and oscillations, the Glim will just be the limit, in the ordinary sense, of its expectation function. This means that in dealing with limit of oscillatory function it is essential to determine its expectation point. Let us now go back to our original problem. Let

(5) g(x) = exp(−1/x2)/xk and h(x) = { cosm1/x2},

where the braces indicate that the set-limit of h(x) is set-valued as $x \to 0$. The product function g(x)h(x) is now a wild oscillation as a product with one factor h(x) wild oscillation (note that an ordinary function multiplied by a wild oscillation and rapid oscillation becomes wild and rapid oscillations, respectively). The Glim of the product function {g(x)h(x)} is given by,

(6) Glim ({g(x)h(x)}) = lim E({g(x)h(x)}), as $x \to 0$,

where E refers to expectation. If g(x) is well-defined and {h(x)} is set-valued then

(7) E({g(x)h(x)}) = g(x)E({h(x)}) [3].

Applying this to (6) we have,

(8) Glim ({g(x)h(x)}) = lim g(x) E({h(x)}), $x \to 0$,

since E(g(x)) is single valued. Therefore,

(9) Glim ({g(x)h(x)}) = lim g(x) lim E({h(x)}), $x \to 0$.

Equation (9) is our basis for solving limits of functions of the forms (5).
It can be proved by repeated use of L'Hospital's rule that lim g(x) = 0, as $x \to 0$; however, when this is multiplied by the wild oscillation {h(x)} L'Hospital's rule fails because the product function has countably infinite zeros in any neighborhood of the origin. At any rate, since both functions,

(10) sinn(1/x2)cosm(1/x2) and sinn(1/x2) + (cosm1/x2),

have well-defined expectation points at the origin, we have now proved this theorem:
Theorem.

(a) Glimexp(−1/x2)[sinn(1/x2)(cosm1/x2)]/xk= 0, as $x \to 0$,

(b) Glim exp(−1/x2)[sinn(1/x2) + (cosm1/x2)]/xk = 0, as $x \to 0$,

for all values of n and m.
However, the Glim of oscillatory function need not be 0, e.g.,

(11) Glim e−x (sinn (1/x2) + cosm(1/x2)),

is obviously not 0. Moreover, the derivative of a function may be set valued. In fact, if a function is oscillatory at a point its derivative is also oscillatory there. For example, it is assumed in [20] that,

(12) f(x) = x2sin(1/x2),

has derivative f′ at 0, and that f′(0) = 0, since both one-sided derivatives satisfy,

(13) $|\Delta f/\Delta x| \le |x2/x| \le x$,

so that

(14) lim $|\Delta f/\Delta x|$ = 0, as $x \to 0+$;

f(x) in this case is differentiable outside the origin but f′(x) has essential discontinuity at the origin. In fact, we have, at $x \neq 0$,

(15) f′(x) = 2xsin(1/x2) − (2/x)cos(1/x2),

which is set-valued. We denote the generalized derivative of a set valued function {f(x)} by GD{f(x)}.

The graph of f(x) is shown in Figure 3. and it is clear from its graph that it has wild oscillation at x = 0 (in fact, it is clear that the set limit of f′(x) is the set of real numbers (–∞,∞) [3]). Note that the term 2/x)cos(1/x2) in (15) oscillates through all values of the real line but the term 2xsin(1/x2) vanishes. This means that the oscillation defined by f′(x) at x = 0 only depends on the term (2/x)cos(1/x2). Thus, the expectation point of the oscillation there is the expectation point of the oscillation (2/x)cos(1/x2. Using the above definition, we have,

(16) GD({x2 sin(1/x2)}) = E({(2/x)cos(1/x2)}).

In view of the symmetry of the wild oscillation,

(18) {f′(x)} = (2/x)cos(1/x2)

from −∞ to +∞ near the origin the right side of (19) tends to 0, as x → 0+. Therefore,

(19) GD({x2 sin(1/x2)}) = 0 at x = 0.

VII. THE GENERALIZED INTEGRAL AS DUAL OF SCHWARTZ DISTRIBUTION

Just as in the space of generalized curves [5] where a function (conventional curve) is not as important in itself as its effect on curvilinear integral of functions along it, in Schwartz distributions [6,28] a function is not so important in itself as its effect on other functions. The effect of a function f(x) defined in the interval [0,1] say, on another such function η(x), is measured by the expression,

(1)∫[0,1] f(x)η(x)dx = Tf(η).

In Schwarz distributions, the focus is on the effect of function f, not on all functions, but only on "civilized" functions η [6]. A function η is civilized if it is infinitely differentiable and vanishes identically at its two ends 0 and 1. The function η belongs to a class of "test functions" which can be substituted for η in Tf(η) of (1) consisting of (a) η itself, (b) a larger class of infinitely differentiable functions ς which vanish at their two ends 0 and 1, (c) the subclass of (b) consisting of the functions sin(2πnx), 1 – cos(2πnx), n = 1, 2, …; (d) a totally different class consisting of continuous piecewise linear functions ξ which vanish identically in the neighborhoods of the two ends; (e) the subclass of (c) consisting of what are called "stump shaped" functions defined as follows: a linear function σ(x) is termed stump shaped if its slope is 1 and – 1, respectively, in two mutually exclusive closed intervals of equal length [a,a + h), (b – h,b], interior to [0,1], constant in the closed interval [a + h,b – h] and vanishes outside the open interval (a,b).

§Figures 1 – 4 were drawn by Noel E. Escultura, Professor of Fine Arts, University of Sto. Tomas, Manila.

VIII. REFERENCES

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